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Actions of Solvable Lie Algebras on Rings with No Nilpotent Elements

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Let R be an algebra with no non-zero nilpotent elements acted on by a finite dimensional solvable restricted Lie algebra L. We examine the relationship between R and the ring of constants R^L. In particular, we prove:

(1) if R^L satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree $p^n d$, where n is the dimension of L;

(2) R is Goldie if and only if R^L is Goldie;

(3) in the case where both R and R^L are Goldie, both R and R^L have the same Goldie rank and the Goldie localization of R can be obtained by inverting the regular elements of R^L. © 1990 Academic Press, Inc.

SECTION 1. INTRODUCTION, DEFINITIONS, AND BASIC REDUCTIONS

Let L be a Lie algebra over a field K of characteristic $p>0$. We say L is *restricted* if, in addition to the usual structure, there is a pth power map $x \mapsto x^{[p]}$ satisfying

(R1) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$

(R2) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, where $is_i(x, y)$ is the coefficient of λ^{i-1} in $(ad(\lambda x+y))^{p-1}(x)$

(R3) $(ady^{[p]})(x) = (ady)^p(x)$ for all $\alpha \in K$, $x, y \in L$.

If R is a K algebra, let $\mathrm{Der}_K(R)$ denote the set of K linear derivations of R. We say that L *acts on* R if there is a Lie algebra homomorphism $\phi\colon L \to \mathrm{Der}_K(R)$ which preserves pth powers, that is $\phi(x^{[p]}) = \phi(x)^p$, for all $x \in L$.

The *ring of constants* of the action of L on R is $R^L = \{r \in R \mid \delta(r) = 0$, all $\delta \in \phi(L)\}$. In [B 88] we showed that if R is non-nilpotent and if L is finite

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dimensional nilpotent restricted then $R^L \neq 0$. It easily follows from that result that if R has no non-zero nilpotent elements and if L is finite dimensional solvable restricted then $R^L \neq 0$. Our goal is to examine the relationship between R and R^L in this case.

We begin by making some basic reductions. The first main result we would like to prove is that if R^L satisfies a polynomial identity of degree d then R satisfies a polynomial identity of degree $p^n d$, where $n = \dim_K L$. Our second main result is that R is Goldie if and only if R^L is Goldie. Suppose we attempt to prove each of these results by induction. If $M \neq 0$ is a proper restricted Lie ideal of L then M acts on R and the quotient restricted Lie algebra L/M acts on R^M with $R^L = (R^M)^{L/M}$. Since R^M has no non-zero nilpotent elements and both M and L/M are solvable with $\dim_K M + \dim_K L/M = \dim_K L$, our results would follow by induction. Therefore, without loss of generality, we may assume that L has no non-zero proper restricted Lie ideals.

Now let N and M be Lie ideals of L and let $\bar{N}$ be the linear span over K of all elements of L of the form $x^{[p^t]}$, $x \in N$ and $t \geqslant 0$. By (R3) it follows that $[x^{[p^t]}, y] \in [N, M]$ for $x \in N$ and $y \in M$, hence $[\bar{N}, M] = [N, M]$. Furthermore, by (R2) and (R3), $(\sum \alpha_i x_i^{[p^{t_i}]})^{[p]} \in \bar{N}$, for all $\alpha_i \in K$, $x_i \in N$, and $t_i \geqslant 0$. As a result, $\bar{N}$ is the smallest restricted Lie ideal of L containing N.

We now consider $\overline{[L, L]}$; since L has no non-zero proper restricted Lie ideals either $\overline{[L, L]} = L$ or $\overline{[L, L]} = 0$. However, by the solvability of L and the above argument, if L is not abelian then $L = \overline{[L, L]}$ and we have $[L, L] = [\overline{[L, L]}, \overline{[L, L]}] = [[L, L], \overline{[L, L]}] = [[L, L], [L, L]] \neq [L, L]$, a contradiction. Hence L must be abelian.

We must split our analysis of L into two cases, but we first must introduce restricted enveloping algebras. Since L is restricted, we can form the *restricted enveloping algebra* $u(L) = U(L)/I$, where $U(L)$ is the *universal enveloping algebra* and I is the ideal of $U(L)$ generated by all $x^p - x^{[p]}$ with $x \in L$. If B is an ordered basis for L, then it follows by Jacobson's Theorem [J 62; Chap. 5, Theorem 12] that $u(L)$ has a K basis consisting of all monomials of the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ with $x_i \in B$, $x_1 < x_2 < \cdots < x_n$, and $0 \leqslant a_i \leqslant p-1$. If $u(L)$ is semisimple then a great deal is known about the relationship between R and R^L. Of particular use to us is the result of Bergen and Cohen [BC 86].

THEOREM 1.1 [BC 86]. *Suppose L is an n-dimensional restricted Lie algebra acting on a semiprime ring R such that $u(L)$ is semisimple. Then*

(1) *if R^L satisfies a polynomial identity of degree d then R satisfies a polynomial identity of degree $p^n d$*

(2) *R is Goldie if and only if R^L is Goldie.*

We now return to our analysis of L. Suppose we are in the case that $x^{[p]} \neq 0$, for every $0 \neq x \in L$. Since L is abelian and has no proper restricted ideals, it follows that L equals the linear span over K of $\{x^{[p^i]} \mid i \geq 0\}$, for any $0 \neq x \in L$. Since L is finite dimensional, if $0 \neq x \in L$ then there exists some minimal $t \geq 0$ such that

$$x^{[p^{t+1}]} = \alpha_t x^{[p^t]} + \alpha_{t-1} x^{[p^{t-1}]} + \cdots + \alpha_s x^{[p^s]},$$

where $\alpha_i \in K$ and $\alpha_s \neq 0$. Now let $y = x^{[p^s]}$ and $k = t - s$ then

$$y^{[p^{k+1}]} = \alpha_t y^{[p^k]} + \cdots + \alpha_s y.$$

By our choice of t, $\{y, y^{[p]}, ..., y^{[p^k]}\}$ is a basis for L over K. Furthermore, since $\alpha_s \neq 0$, $\{y^{[p]}, y^{[p^2]}, ..., y^{[p^{k+1}]}\}$ is also a basis for L over K.

Let $\{\Delta_i\}$ be a set of basis monomials for $u(L)$ over K using the ordered basis $y < y^{[p]} < \cdots < y^{[p^k]}$ for L over K. Then $\{\Delta_i^p\}$ is also a set of basis monomials for $u(L)$ over K in terms of the ordered basis $y^{[p]} < y^{[p^2]} < \cdots < y^{[p^{k+1}]}$ for L over K. Suppose $w = \sum \alpha_i \Delta_i \in u(L)$ with $\alpha_i \in K$, is a non-zero element of $u(L)$. Then, since L is abelian, $w^p = \sum \alpha_i^p \Delta_i^p$ is also non-zero as some $\alpha_i^p \neq 0$. Therefore $u(L)$ is a finite dimensional algebra over K having no non-zero nilpotent elements, thus $u(L)$ is semisimple. We refer to this situation as the separable case and note that in this case the desired results relating R and R^L hold even without assuming that R has no non-zero nilpotent elements.

The only remaining case to consider is where L contains an element $x \neq 0$ such that $x^{[p]} = 0$. In this case $\{\alpha x \mid \alpha \in K\}$ is a restricted Lie ideal of L, hence $L = \{\alpha x \mid \alpha \in K\}$ and if we let $\delta = \phi(x)$ then $R^L = \{r \in R \mid \delta(r) = 0\}$ and $\delta^p = 0$. This situation is referred to as the inseparable case and, in light of our series of reductions, it is the inseparable case which requires the real work in this paper.

We have seen, in the separable case, that the assumption that R has no non-zero nilpotent elements is not necessary. However, the following example, based on an example on group actions by Bergman and Kharchenko, illustrates the importance of that assumption when studying the action of solvable Lie algebras.

EXAMPLE. Let $S = K[x, y]$ be the non-commutative free algebra in two variables over a field K of characteristic $p > 0$. Let $R = M_2(S)$ be the 2×2 matrices over S and let L be the inner derivations of R induced by the elements of the set

$$\left\{ \begin{bmatrix} \alpha & \beta x + \gamma y \\ 0 & 0 \end{bmatrix} \bigg|_{\alpha, \beta, \gamma \in K} \right\}.$$

L is easily seen to be a three-dimensional solvable restricted Lie algebra spanned by the inner derivations induced by $\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0 & x\\ 0 & 0\end{smallmatrix}\right]$, and $\left[\begin{smallmatrix}0 & y\\ 0 & 0\end{smallmatrix}\right]$. A direct calculation shows that $R^L = \{\left[\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right] |_{\alpha \in K}\}$, thus R^L is Goldie and satisfies a polynomial identity, whereas R is neither Goldie nor satisfies a polynomial identity.

Among the most interesting and difficult results in the study of finite groups acting on rings is the result of Kharchenko [K 75], which states that if a ring R with no non-zero nilpotent elements is acted on by any finite group G then the fixed ring R^G is non-zero. It can then by shown that if R^G satisfies a polynomial identity then so does R and R is Goldie if and only if R^G is Goldie. In light of these results on group actions, it is conceivable that the results we prove on Lie algebra actions hold even if the Lie algebra is not solvable. This is a question we would like to examine in the future, however, at present it is not even known if $R^L \neq 0$ whenever a finite dimensional restricted Lie algebra L acts on an algebra R with no non-zero nilpotent elements.

Section 2. Polynomial Identities

The goal of this section is to prove that if R has no non-zero nilpotent elements and if L is an n-dimensional solvable restricted Lie algebra such that R^L satisfies a polynomial identity of degree d then R satisfies a polynomial identity of degree $p^n d$. In light of our reductions in the previous section, it suffices to prove the result when L is one-dimensional and contains some $x \neq 0$ such that $x^{[p]} = 0$.

An ideal P of R is called *completely prime* if R/P is a domain. A well-known result of Andrunakievitch and Rjahubin [AR 68] states that in a ring with no non-zero nilpotent elements, the intersection of the completely prime ideals is zero. We begin with an easy but useful lemma.

Lemma 2.1. *Let P be a completely prime ideal and δ a derivation of an algebra R of characteristic $p > 0$. If $\delta^{p-1}(R) \subseteq P$, then $\delta(R) \subseteq P$.*

Proof. Let t be the smallest positive integer such that $\delta^t(R) \subseteq P$; clearly $p-1 \geqslant t \geqslant 1$. If $t > 1$, let $a, b \in R$ and consider

$$\delta^t(a\delta^{t-2}(b)) = \delta^t(a)\delta^{t-2}(b) + t\delta^{t-1}(a)\delta^{t-1}(b) + \cdots + a\delta^{2t-2}(b).$$

Thus $t\delta^{t-1}(R)\delta^{t-1}(R) \subseteq P$ and since $t < p$ and P is completely prime, it follows that $\delta^{t-1}(R) \subseteq P$, a contradiction. Hence $t = 1$ and $\delta(R) \subseteq P$.

For the remainder of this section we assume that R is an algebra over a field K of characteristic $p > 0$ acted on by a one-dimensional restricted

Lie algebra L possessing an element $x \neq 0$ such that $x^{[p^l]} = 0$ and we let $\delta = \phi(x)$. The following important lemma relies heavily on a result of Smith [S 75] on centralizers of algebraic elements.

LEMMA 2.2. *Suppose R^L is a domain satisfying a polynomial identity of degree d, then R satisfies a polynomial identity of degree pd.*

Proof. We form the differential operator ring $S = R[t; \delta]$; it is a K algebra with the additive structure of the polynomial ring $R[t]$ and multiplication extending that of R and satisfying

$$tr = rt + \delta(r), \qquad (*)$$

for all $r \in R$. We now assume that R contains a unit element, for adjoining a unit element will not affect the hypotheses of this lemma. Since R now contains a unit, $R[t; \delta]$ contains the element t and we can consider $C_{R[t;\delta]}(t)$, the centralizer in $R[t; \delta]$ of t.

Suppose $s = r_0 + r_1 t + \cdots + r_l t^l \in R[t; \delta]$; then, by (*)

$$ts - st = \delta(r_0) + \delta(r_1)t + \cdots + \delta(r_l)t^l.$$

Therefore $s \in C_{R[t;\delta]}(t)$ if and only if each $r_i \in R^L$ and so, $C_{R[t;\delta]}(t)$ is equal to the polynomial ring $R^L[t]$. In particular, $C_{R[t;\delta]}(t)$ is a domain and also satisfies a polynomial identity of degree d.

It is clear that commutation by t is an inner derivation δ of $R[t; \delta]$ which extends the derivation δ from R to $R[t; \delta]$ and $\delta^p = 0$. Let $I \neq 0$ be an ideal of $R[t; \delta]$ and let $k \geqslant 0$ be the largest integer such that $\delta^k(I) \neq 0$. Since $\delta(I) \subseteq I$, it follows that $0 \neq \delta^k(I) \subseteq I \cap C_{R[t;\delta]}(t)$, thus every non-zero ideal of $R[t; \delta]$ intersects $C_{R[t;\delta]}(t)$ non-trivially. Since $C_{R[t;\delta]}(t)$ is a domain, non-zero ideals of $R[t;\delta]$ cannot annihilate each other; hence $R[t; \delta]$ is prime.

As a result t is an element of the prime ring $R[t; \delta]$ whose centralizer satisfies a polynomial identity of degree d. Furthermore, since $R[t; \delta]$ has characteristic $p > 0$, commutation by t^p induces the derivation $\delta^p = 0$; hence t^p belongs to the center of $R[t; \delta]$.

In [S 75], Smith examines prime rings with an element t algebraic over the center, whose centralizer satisfies a polynomial identity of degree d. She proves that the ring must satisfy a polynomial identity of degree at most d times the degree of algebraicity of t. Therefore, $R[t; \delta]$ satisfies a polynomial identity of degree pd, hence R satisfies a polynomial identity of degree pd.

We continue by showing that each completely prime ideal of R must fall into one of two classes.

LEMMA 2.3. *Suppose R^L satisfies a polynomial identity of degree d; then for every completely prime ideal P either R/P satisfies a polynomial identity of degree pd or $\delta(R) \subseteq P$.*

Proof. If P is a completely prime ideal of R, let M be the sum of all the δ invariant ideals of R which are contained in P. Suppose we let $a, b \in R$ such that $ab \in M$ and $\delta(a)$, $\delta(b) \in M$; it then follows that $\delta^i(a)$, $\delta^i(b) \in M$, for all $i \geqslant 1$, and additionally either $a \in P$ or $b \in P$. If $a \in P$, consider the ideal I of R generated by $a, \delta(a), ..., \delta^{p-1}(a)$; I is both δ invariant and contained in P, hence $I \subseteq M$. Thus, if $a \in P$ then $a \in M$ and similarly, if $b \in P$ then $b \in M$.

The action of L on R induces an action of L on the ring $\bar{R} = R/M$. Suppose $\bar{a}$, $\bar{b} \in (\bar{R})^L$ such that $\overline{ab} = \bar{0}$; then the argument in the previous paragraph shows that either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$. Hence $(\bar{R})^L$ is a domain.

We now consider $\delta^{p-1}(R)$; if $\delta^{p-1}(R) \subseteq M$ then $\delta^{p-1}(R) \subseteq P$. Hence, by Lemma 1, $\delta(R) \subseteq P$. On the other hand, suppose $\delta^{p-1}(R) \nsubseteq M$; thus $(\delta^{p-1}(R) + M)/M$ is a non-zero ideal of $(\bar{R})^L$. Since $\delta^{p-1}(R) \subseteq R^L$, it follows that $\delta^{p-1}(R)$ satisfies a polynomial identity of degree d. Clearly, $(\delta^{p-1}(R) + M)/M$ now also satisfies a polynomial identity of degree d and since $(\bar{R})^L$ is a domain, it too satisfies a polynomial identity of degree d. We are now in the situation described in Lemma 2.2; therefore $\bar{R}$ satisfies a polynomial identity of degree pd. Since $R/P \approx (R/M)/(P/M)$, we conclude that either R/P satisfies a polynomial identity of degree pd or $\delta(R) \subseteq P$.

We can now prove the main result of this section. In the proof we use the well-known fact that if a semiprime ring satisfies a polynomial identity of degree m, then it satisfies the standard identity of degree m.

THEOREM 2.4. *Let R be an algebra with no non-zero nilpotent elements acted on by a finite dimensional solvable restricted Lie algebra L. If R^L satisfies a polynomial identity of degree d, then R satisfies a polynomial identity of degree $p^n d$, where n is the dimension of L.*

Proof. By the series of reductions done in the previous section, we may assume that L is one-dimensional and contains some $x \neq 0$ such that $x^{[p]} = 0$. We let $\delta = \phi(x)$ and first consider the special case where $\delta(I) \neq 0$, for every ideal $I \neq 0$ of R. In this case, let A be the intersection of all the completely prime ideals P of R such that $\delta(R) \subseteq P$ and let B be the intersection of the completely prime ideals P of R such that R/P satisfies a polynomial identity of degree pd. By Lemma 2.3, $0 = A \cap B \supseteq AB + BA$ and since $\delta(B) \subseteq \delta(R) \subseteq A$ it follows that $\delta(B^2) \subseteq \delta(B) B + B\delta(B) \subseteq AB + BA = 0$. Therefore $B^2 = 0$ which implies that $B = 0$; hence R is a subdirect sum of domains satisfying a polynomial identity of degree pd. Thus all these domains satisfy the standard identity of degree pd; hence R also satisfies the standard identity of degree pd.

Now let M be the sum of all the ideals of R contained in R^L and let N be the annihilator of M. Since R is semiprime, N is an ideal of R with $M \cap N = 0$. Furthermore, since $\delta(M) = 0$ it follows that $\delta(N) \subseteq N$; therefore L acts on N. If I is an ideal of N such that $\delta(I) = 0$ then NIN is an ideal of R contained in $M \cap N$; hence $I = 0$. Therefore we can apply the previous argument to conclude that N satisfies the standard identity of degre pd.

Since $M \subseteq R^L$, M satisfies a polynomial identity of degree d; hence both M and N satisfy the standard identity of degree pd. Now consider the ideal $M \oplus N$; it is an essential ideal of R satisfying the standard identity of degree pd. Let C be the intersection of all the prime ideals of R not containing $M \oplus N$ and let D be the intersection of the prime ideals containing $M \oplus N$. Since R is semiprime, $CD \subseteq C \cap D = 0$; however $M \oplus N \subseteq D$, thus $C(M \oplus N) = 0$. Since $M \oplus N$ is essential, $C = 0$. Finally, let P be a prime ideal not containing $M \oplus N$; thus $((M \oplus N) + P)/P$ is a non-zero ideal of the prime ring R/P and $((M \oplus N) + P)/P$ satisfies the standard identity of degree pd. Thus, R/P also satisfies the standard identity of degree pd and, since $C = 0$, R is therefore a subdirect sum of rings satisfying the standard identity of degree pd. Hence R satisfies a polynomial identity of degree pd thereby proving the theorem.

SECTION 3. GOLDIE RINGS AND LOCALIZATION

In this section we prove that if R has no non-zero nilpotent elements and is acted on by a finite dimensional solvable restricted Lie algebra L, then R is Goldie if and only if R^L is Goldie. We then show that in the case where both R and R^L are Goldie, the Goldie localization of R can be obtained by inverting the regular elements of R^L. Furthermore, both R and R^L have the same Goldie rank.

Throughout this section we often use two basic facts about rings R with no non-zero nilpotent elements. The first is that if $a, b \in R$ such that $ab^2 = 0$, then $ab = 0$. The second is that the left and right annhilators of any subset T of R agree and form a two-sided ideal of R which we denote as $\mathrm{Ann}(T)$.

Suppose that λ is an essential left ideal of R; then if $\mathrm{Ann}(\lambda) \neq 0$ it would follow that $(\mathrm{Ann}(\lambda) \cap \lambda) \neq 0$, whereas $(\mathrm{Ann}(\lambda) \cap \lambda)^2 = 0$, a contradiction. Thus, $\mathrm{Ann}(\lambda) = 0$ and so, the left singular ideal of R is zero. As a result, when trying to show that a ring with no non-zero nilpotent elements is left (or right) Goldie, it suffices to show that it has no infinite direct sums of left (or right) ideals.

In all that follows in this section, we assume that R is an algebra over a field of characteristic $p>0$ and has no non-zero nilpotent elements.

LEMMA 3.1. *Let δ be a derivation of R such that $\delta^p=0$ and let $M=\mathrm{Ann}(\delta^{p-1}(R))$. Then*

(1) $M=\mathrm{Ann}(\delta(R))$ *and*

(2) $\delta(M)=0$.

Proof. Let t be the smallest positive integer such that $M\delta^t(R)=0$. If $t>1$, let $b\in R$ and consider

$$0=M\delta^t(b\delta^{t-2}(b))=tM\delta^{t-1}(b)\delta^{t-1}(b).$$

Since $t<p$ we have $M\delta^{t-1}(b)^2=0$; hence $M\delta^{t-1}(b)=0$. Therefore $t=1$; hence $M\delta(R)=0$ and $M=\mathrm{Ann}(\delta(R))$.

Since M is invariant under the action of δ, we have $\delta(M)\subseteq M\cap\delta(R)=0$.

The next lemma is used throughout this section.

LEMMA 3.2. *If $M=\mathrm{Ann}(T)$, for some $T\subseteq R$, then R is left (or right) Goldie if and only if R/M and M are both left (or right) Goldie*

Proof. Since M is an ideal of R, $M=\mathrm{Ann}(RTR)$; thus without loss of generality, we may assume that T is an ideal of R. Suppose R is left Goldie; then any ideal I of R, when viewed as a ring, would be left Goldie, for any infinite direct sum of left ideals of I, $\oplus\beta_i$, would contain an infinite direct sum of left ideals of R, $\oplus I\beta_i$. In particular, M is left Goldie and we turn our attention to R/M. If $a\in R$ such that $a^2\in M$, then $Ta^2=0$; hence $Ta=0$ and $a\in M$. Therefore R/M has no non-zero nilpotent elements. Now suppose that R/M has an infinite direct sum of left ideals $\oplus\lambda_i/M$, where each λ_i is a left ideal of R properly containing M. Consider the left ideals $T\lambda_i$; clearly $0\neq T\lambda_i\subseteq\lambda_i$ and we claim that the sum of the $T\lambda_i$ is direct. If not, then by reordering the $T\lambda_i$ we have $T\lambda_1\cap(T\lambda_2\oplus\cdots\oplus T\lambda_m)\neq 0$ and since the sum of the λ_i/M is direct, it follows that $((T\lambda_1\cap(T\lambda_2\oplus\cdots\oplus T\lambda_m))+M)/M=0$. Therefore $T\lambda_1\cap(T\lambda_2\oplus\cdots\oplus T\lambda_m)\subseteq M\cap T=0$; hence $\oplus T\lambda_i$ is an infinite direct sum of left ideals of R, contradicting the fact that R is left Goldie. Thus R/M is left Goldie.

Conversely, suppose that R/M and M are both left Goldie and that $\oplus\lambda_i$ is an infinite direct sum of left ideals of R. We note that only a finite number of the λ_i are contained in M, otherwise M would not be left Goldie. Therefore, by deleting those λ_i which are contained in M, we may assume that none of the λ_i in our infinite direct sum are contained in M. Therefore $\oplus T\lambda_i$ is also an infinite direct sum of left ideals of R. However, since R/M

is left Goldie, by reordering the $T\lambda_i$ we obtain $((T\lambda_1+M)/M)\cap(((T\lambda_2+M)/M)\oplus\cdots\oplus((T\lambda_m+M)/M))\neq 0$. As a result there exists $a_i\in T\lambda_i$ such that $0\neq a_1+a_2+\cdots+a_m\in M$, but since each $a_i\in T$ we have $a_1+\cdots+a_m\in M\cap T=0$, a contradiction. Thus R must be left Goldie.

We can now prove the first main result of this section.

THEOREM 3.3. *Let R be an algebra with no non-zero nilpotent elements acted on by a finite dimensional solvable restricted Lie algebra L. If R^L is left (or right) Goldie then R is left (or right) Goldie.*

Proof. In order to prove that R is left Goldie, we apply our reductions from Section 1; therefore we may assume that L is one-dimensional and contains some $x\neq 0$ such that $x^{[p]}=0$. Let $\delta=\phi(x)$ and $M=\mathrm{Ann}(\delta^{p-1}(R))$; by Lemma 3.1, $M\subset R^L$ and we may therefore consider the ring R^L/M. Suppose $a\in R$ such that $a\delta^{p-1}(R)\subseteq M$; then $a\delta^{p-1}(R)\delta^{p-1}(R)=0$, hence $a\delta^{p-1}(R)=0$, and so $a\in M$. Therefore the ideal $(\delta^{p-1}(R)+M)/M$ of R^L/M is essential. As in the proof of Lemma 3.2, R^L/M is Goldie with no non-zero nilpotent elements, thus $(\delta^{p-1}(R)+M)/M$ contains a regular element $\delta^{p-1}(b)+M$. Now consider the map

$$\psi\colon R\to\bigoplus_{i=0}^{p-1}R^L$$

defined as

$$\psi(r)=\bigoplus_{i=0}^{p-1}\delta^{p-1}(\delta^i(r)b).$$

ψ is clearly a left R^L module homomorphism and we now examine its kernel. If $\psi(r)=0$ then, letting $i=p-1$, we have $\delta^{p-1}(r)\delta^{p-1}(b)=0$. Since $\delta^{p-1}(r)\in R^L$ and $\delta^{p-1}(b)$ is regular modulo M, it follows that $\delta^{p-1}(r)\in M$. However, by Lemma 3.1, $M=\mathrm{Ann}(\delta(R))$; thus $\delta^{p-1}(r)\in\delta(R)\cap\mathrm{Ann}(\delta(R))=0$. Now letting $i=p-2$ and using the fact that $\delta^{p-1}(r)=0$, we obtain $\delta^{p-2}(r)\delta^{p-1}(b)=0$ and the same argument as before results in $\delta^{p-2}(r)=0$. By continuing to let $i=p-3, p-4, ..., 1$ we eventually see that $\delta(r)=0$. Finally letting $i=0$ results in $r\delta^{p-1}(b)=0$; hence $r\in M$. Since $M\subseteq R^L$ and $M\delta^{p-1}(b)=0$, it follows that M is the kernel of ψ.

Since R^L is left Goldie, $\bigoplus_{i=0}^{p-1}R^L$ has no infinite direct sums of left R^L submodules. Furthermore since ψ is a left R^L module homomorphism with kernel M, it follows that the ring R/M has no infinite direct sums of R^L submodules. In particular, R/M can have no infinite direct sums of left ideals. However, as in the proof of Lemma 3.2, R/M has no non-zero nilpotent elements; hence R/M is left Goldie. Finally, M is an ideal of R^L; hence, by Lemma 3.2, M is left Goldie and therefore, again by Lemma 3.2, we conclude that R is left Goldie.

We conclude this paper with our final main result. However, we should first mention the related result of Popov [P 83], where it is shown that R is Goldie if and only if R^L is Goldie provided that R is prime and L is any finite dimensional restricted Lie algebra acting on R as outer derivations.

THEOREM 3.4. *Let R be an algebra with no non-zero nilpotent elements acted on by a finite dimensional solvable restricted Lie algebra L. Then*

(1) *R is left (or right) Goldie if and only if R^L is left (or right) Goldie.*

Furthermore, in the case where both R and R^L are left (or right) Goldie

(2) *$Q(R)=R_T$, where $Q(R)$ is the Goldie localization of R and R_T is the localization of R at the regular elements T of R^L*

(3) *$Q(R)^L=Q(R^L)$ and*

(4) *R and R^L have the same Goldie rank.*

Proof. Suppose R is left Goldie and let $b^{-1}a\in Q(R)$, where $a,b\in R$ with b regular. By the left Ore condition there exist $c,d\in R$ with d regular, such that $cb=da$, thus $d^{-1}c=ab^{-1}$. If $(b^{-1}a)^2=0$ then it follows that $0=ab^{-1}a=d^{-1}ca$; therefore $ca=0$ and we then also have $0=cba=da^2$. Hence $a^2=0$ and finally $a=0$. As a result, $Q(R)$ is an Artinian ring with no non-zero nilpotent elements; therefore $Q(R)=D_1\oplus D_2\oplus\cdots\oplus D_m$ for division rings D_i. In particular, R has Goldie rank m.

The action of L extends to an action on $Q(R)$ via $\delta(b^{-1})=-b^{-1}\delta(b)b^{-1}$, for all $\delta\in\phi(L)$ and b regular in R. Each D_i is an ideal of $Q(R)$ with $\delta(D_i)=\delta(D_i^2)\subseteq\delta(D_i)D_i+D_i\delta(D_i)$, for every $\delta\in\phi(L)$; hence the action of L also restricts to an action on D_i. Now let $R_i=D_i\cap R$; each R_i is an L invariant ideal of R and $R_1\oplus R_2\oplus\cdots\oplus R_m$ is then an essential L invariant ideal of R. If $b^{-1}a\in Q(R)$ let $c\in R_1\oplus R_2\oplus\cdots\oplus R_m$ be regular, thus $b^{-1}a=(cb)^{-1}(ca)\in Q(R_1\oplus\cdots\oplus R_m)$. Thus the regular elements of $R_1\oplus\cdots\oplus R_m$ are an Ore set for R and $Q(R)=Q(R_1\oplus\cdots\oplus R_m)=Q(R_1)\oplus\cdots\oplus Q(R_m)$, and so $D_i=Q(R_i)$. In particular, each R_i must be an Ore domain.

Let $\lambda\neq 0$ be a left ideal of R_i; since R_i is an Ore doman, λ is essential in R_i. However, Lemma 4.2 of [BM 86], which is based on an argument of Popov [P 83], states that whenever a non-singular ring is acted on by a finite dimensional restricted Lie algebra L then every essential left ideal λ contains an L invariant essential left ideal $\bar{\lambda}$. Since L is solvable, $0\neq\bar{\lambda}^L=\bar{\lambda}\cap R_i^L\subseteq\lambda\cap R_i^L$; therefore λ intersects R_i^L non-trivially. Now let $a\neq 0$, $b\neq 0\in R_i$, and $\beta=\{r\in R_i\mid rb\in R_ia\}$; β is a non-zero left ideal of R_i thus $\beta\cap R_i^L\neq 0$. Therefore there exist $0\neq c\in R_i^L$ and $0\neq d\in R_i$ such that $cb=da$. In particular, if $a\in R_i^L$ then it follows that T_i, the non-zero elements of R_i^L, is an Ore set for R_i. Furthermore, if $a,b\in R_i^L$ then $d\in R_i^L$ and hence R_i^L is an Ore domain. In addition, if $0 \neq b \in R_i$ then $R_i b \cap R_i^L \neq 0$; thus letting $0 \neq cb \in R_i b \cap R_i^L$ we have $b^{-1} = (cb)^{-1} c \in (R_i)_{T_i}$, the localization of R_i at T_i. Hence $(R_i)_{T_i} = Q(R_i) = D_i$. Now if $q \in D_i^L$ then $q = b^{-1} a$, where $b \in R_i^L$ and $a \in R_i$; therefore $0 = \delta(q) = b^{-1}\delta(a)$ for all $\delta \in \phi(L)$. Thus $a \in R_i^L$ and hence $D_i^L = Q(R_i^L)$.

Since each R_i^L is an Ore domain, $A = R_1^L \oplus \cdots \oplus R_m^L$ is a left Goldie ring. Additionally, A is an essential left ideal of R^L, thus any infinite direct sum of left ideals of R^L would intersect A in an infinite direct sum of left ideals of A. Therefore R^L is left Goldie and in light of Theorem 3.3, R is left Goldie if and only if R^L is left Goldie.

Now, when R and R^L are both left Goldie we combine our above results to obtain

$$\begin{aligned} Q(R^L) &\supseteq Q(R_1^L \oplus \cdots \oplus R_m^L) \\ &= Q(R_1^L) \oplus \cdots \oplus Q(R_m^L) = Q(R_1)^L \oplus \cdots \oplus Q(R_m)^L \\ &= D_1^L \oplus \cdots \oplus D_m^L = (D_1 \oplus \cdots \oplus D_m)^L = Q(R)^L \supseteq Q(R^L). \end{aligned}$$

In particular, $Q(R^L) = Q(R)^L$ and therefore both R and R^L have Goldie rank m. Finally, since each $Q(R_i)$ can be obtained by inverting elements in T_i, $Q(R)$ can be obtained by inverting elements in $T_1 \oplus \cdots \oplus T_m$ and we have $Q(R) = R_T$.

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